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UNSTEADY FLOWS OF AN INHOMOGENEOUS INCOMPRESSIBLE VISCOUS FLUID

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This paper deals with a theoretical analysis of the transfer of reactive impurities by open and filtration flows of an incompressible viscous fluid. The first section of the paper studies the model of an inhomogeneous incompressible viscous fluid, which is widely used in meteorology and oceanology, with additional allowance for the drag of the magnetic field or porous medium. Another object of research in this paper is the model of filtration of an inhomogeneous incompressible fluid in porous media proposed by V. N. Monakhov (1977) (Section 2). In both models, hydrodynamic flows determine the motion of the mixture as a whole and the temperature and concentration distributions of the components of an inhomogeneous fluid are described by a common nonlinear system of equations of diffusive heat and mass transfer.

Key words: *viscous fluid, impurity, temperature, existence theorem.*

1. Diffusive Model of an Inhomogeneous Incompressible Viscous Fluid. 1.1. *Equations of the Model.* The plane flows of inhomogeneous fluids are described by the following Navier–Stokes type equations for the velocity $\boldsymbol{u}$, pressure p, and density ρ of the mixture [1]:

$$\rho\Big(\frac{d\boldsymbol{u}}{dt}-\gamma\boldsymbol{u}\Big)-\mu\Delta\boldsymbol{u}+\nabla p=\rho\boldsymbol{f},\qquad \nabla\cdot\boldsymbol{u}=0;\qquad \frac{d\rho}{dt}=0. \tag{1}$$

Here $d/dt=\partial/\partial t+(\boldsymbol{u}\cdot\nabla)$, $\mu=\text{const}>0$ is the dynamic viscosity, and $\rho\boldsymbol{f}$ is the external-force vector. The last equation in (1) is the fluid incompressibility condition, and the relation $\nabla\cdot\boldsymbol{u}=0$ is a consequence of this condition and the flow continuity equation $\rho_t+\nabla\cdot\rho\boldsymbol{u}=0$.

The term $\rho\gamma\boldsymbol{u}$ in (1) simulates the drag force of the magnetic field [2] or the porous medium (Joukowski model [3, p. 23]). The coefficient $\gamma=\gamma(x,\boldsymbol{s})$ is a specified function of the coordinates $x=(x_1,x_2)$ and the vector $\boldsymbol{s}=(s_0,\ldots,s_m)$, whose components are the temperature s_0 and the concentrations of the mixture components $s_i=\rho_i/\rho$, $i=\overline{1,m}$ (ρ_i are the densities of the components).

The nonlinear equations of convective diffusion for $\boldsymbol{s}=(s_0,\ldots,s_m)$ are written as follows [4, 5]:

$$\rho\,\frac{ds_i}{dt}-\nabla\cdot\sum_{j=0}^{m}\lambda_{ij}\nabla s_j=h_i(x,\rho,\boldsymbol{s}),\qquad i=\overline{0,m}. \tag{2}$$

Here h_i ($i=\overline{1,m}$) are the chemical-reaction rates, $h_0=\displaystyle\sum_{i=1}^{m}c_ih_i$ is the potential of the heat sources, and $\boldsymbol{q}_i=-\displaystyle\sum_{j=0}^{m}\lambda_{ij}\nabla s_j$ ($i=\overline{0,m}$) are diffusion flows [5]. Often, h_i ($i=\overline{1,m}$) are expressed in divergent form in terms of the chemical potentials φ_i: $h_i=\nabla\cdot\varphi_i$ [4, 5].

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We note that for $\gamma \equiv 0$ and $\partial \boldsymbol{f}/\partial s_i = 0$, the diffusion equations (2) are separated from system (1) and are solved after finding the velocity $\boldsymbol{u}$, density ρ, and pressure p.

According to [5], the diffusion matrix $D = \{\lambda_{ij}\}$ $[(i,j) = \overline{1,m}\,]$ should obey the physically justified equality $\sum_{i=1}^{m} \boldsymbol{q}_i = 0$ $(\det D = 0)$. The conditions of chemical equilibrium for the diffusion process lead to the relation $\sum_{i=1}^{m} h_i = 0$ [4]. Since for $s_k \ll 1$ the effect of the remaining components s_l $(l \neq k)$ on the propagation of the kth impurity is negligible, we have $(\lambda_{ij}, h_i)|_{s_k=0} = 0$ $(i \neq j$ and $k = \overline{1,m}\,)$.

Thus, the coefficients (λ_{ij}, h_i) of Eqs. (2) obey the following assumptions: (i) $\sum_{i=1}^{m} \lambda_{ij} = \lambda_{ij}\Big|_{s_k=0} = 0$ $(j \neq i)$; $\sum_{i=1}^{m} h_i = h_l\Big|_{s_k=0} = 0$, $(l,k) = \overline{1,m}$. As will be shown later, conditions (i) provide the validity of the necessary relation $\sum_{i=1}^{m} s_i = 1$. In addition, after the determination of the solution $\boldsymbol{s}(x,t)$ of system (2), the first condition of (i) allows one to find the velocities $\boldsymbol{u}_i$ of the mixture components from the formulas [4]: $\boldsymbol{u}_i = \boldsymbol{u} + \boldsymbol{q}_i \rho_i^{-1}$ $(i = \overline{1,m}\,)$. The indicated properties are advantages of the adopted form (i) of diffusion flows. Generally speaking, the widely used simplified form of diffusion flows in the form of Fick's laws $(-\boldsymbol{q}_i = d_i \nabla s_i$, $i = \overline{1,m}\,)$ does not ensure that the relation $\sum_{i=1}^{m} s_i = 1$ is satisfied, except for $d_i = d$ $(i = \overline{1,m}\,)$ as is the case for binary mixtures [4, 5].

In the present paper, satisfaction of the following additional conditions is required: (ii) the augmented diffusion matrix $D_0(\delta) = \{\lambda_{ij}\}$ $[(i,j) = \overline{0,m}\,]$ is quasitriangular, i.e., $\sup|\lambda_{ij}| = \delta \ll 1$ for $j > i$, $\lambda_{ii} \geq d > 0$, $i = \overline{0,m-1}$, and $\lambda_{mj} = -\lambda_{m-1m-1}$, $j = \overline{1,m-1}$.

According to these conditions, the matrix $D_0(0)$ is triangular. Boyarskii [6] proved that a steady-state diffusion process with a general regular diffusion matrix $D_0 = \{\lambda_{ij}\}$ $[(i,j) = \overline{0,m}\,]$ can be reduced to a diffusion model with a quasitriangular matrix $D_0(\delta) = R_\delta D_0 R_\delta^{-1}$ $\forall\ \delta \ll 1$, where R_δ is an invertible matrix. Therefore, in the steady-state case, conditions (ii) can be assumed to be satisfied automatically.

1.2. *Initial-Boundary-Value Problem.* Let $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary $\partial\Omega \subset C^{2+\alpha}$ $(\alpha > 0)$, $Q = \Omega \times (0,T)$, (γ_i^1, γ_i^2) $(i = \overline{1,l}\,)$ are adjacent arcs on $\partial\Omega$, $\Gamma_i^k = \gamma_i^k \times (0,T)$ $(k = 1,2,\ i = \overline{1,l}\,)$, $\Gamma^k = \cup_1^l \Gamma_i^k$, $\Omega_0 = \{x \in \Omega,\ t = 0\}$, $\partial_0 Q = \Gamma \cup \Omega_0$, $\Gamma = \Gamma^1 \cup \Gamma^2$, and $\Gamma_0 = \Gamma \cap \Omega_0$. We consider the following initial-boundary-value problem:

$$(\boldsymbol{u} - \boldsymbol{U})_{\partial_0 Q} = (\rho - \rho^0)_{\Omega_0} = 0; \qquad \boldsymbol{U}\Big|_{\Gamma} = (\nabla \cdot \boldsymbol{U})_Q = 0; \tag{3}$$

$$(\boldsymbol{s} - \boldsymbol{S})_{\Omega_0 \cup \Gamma^1} = (\nabla s_k \cdot \boldsymbol{n} - G_k)_{\Gamma^2} = 0, \qquad k = \overline{0,m}. \tag{4}$$

Here $\boldsymbol{U}(x,t)$, $\boldsymbol{S}(x,t)$, $\boldsymbol{G}(x,t,\rho,\boldsymbol{s}) = (G_0, \ldots, G_m)$ are the continuations in Q for the vectors specified on $\partial_0 Q$, and, from physical considerations, we have $S_i \geq 0$ $(i = \overline{1,m}\,)$, $\sum_{i=1}^{m} S_i = 1$, and $0 < r^0 \leqslant \rho^0(x) \leqslant r^1 < \infty$ $(x \in \Omega)$.

It is assumed that the vectors $(\gamma, \lambda_{ij}, h_i, \boldsymbol{f})(x, \boldsymbol{s})$ of the coefficients of Eqs. (1) and (2) and the boundary functions $(\rho^0, \rho_t^0, \nabla\rho^0)(x)$, $(\boldsymbol{U}, \boldsymbol{S})(x,t)$, and $\boldsymbol{G}(x,t,\rho,\boldsymbol{s})$ in (3) and (4) are uniformly continuous after Hölder with respect to all arguments:

$$\|(\gamma, \lambda_{ij}, \nabla\lambda_{ij}, h_i, \rho^0, \rho_t^0, \nabla\rho^0, \boldsymbol{f}, \boldsymbol{U}, \boldsymbol{S}, \boldsymbol{G})\|_{C^\alpha(E)} = M, \qquad \alpha > 0. \tag{5}$$

Here $E = \{(x,t) \in Q,\ \rho \in (r^0, r^1),\ s_0 \in (\theta^0, \theta^1),\ s_k \in (0,1),\ k = \overline{1,m}\,\}$ (θ^k, where $k = 0, 1$, are specified constants).

We note that assumptions (5) ensure, in particular, that zero order matching conditions are satisfied for problem (1)–(4). In addition to assumptions (i) on coefficients (2), we require satisfaction of the following relations for the boundary functions in (4):

$$\sum_{k=1}^{m} G_k\Big|_{\Gamma^2} = G_i\Big|_{s_j=0} = 0, \qquad (i,j) = \overline{1,m}. \tag{6}$$

1.3. *Regularized Problem.* We continue the vectors $(\gamma, \lambda_{ij}, h_k, \boldsymbol{f})$ of the coefficients of Eqs. (1) and (2) and $(\rho^0, \boldsymbol{U}, \boldsymbol{S}, \boldsymbol{G})$ of the boundary functions in (3) and (4), which are specified on the set $E = Q \times (r^0, r^1) \times (\theta^0, \theta^1) \times (0,1)^m$, into the entire space $\mathbb{R}^n$ $(n = m+5)$: over the variables $(\rho, \boldsymbol{s})$, the continuation is performed up to the extreme values of these components, and over (x,t) with preservation of smoothness up to functions that are finite for $|(x,t)| \gg 1$.

We introduce the Steklov averaging operation for the functions $f(y), y \in \mathbb{R}^n$ $(n \leqslant m+5)$:

$$f(y,\varepsilon) = \varepsilon^{-n} \int\limits_{|z-y|<\varepsilon} \omega(|z-y|\varepsilon^{-1}) f(z)\, dz \equiv R^n_\varepsilon(f \mid y), \qquad \varepsilon > 0.$$

Here $\omega(\xi)$ is a smooth function (averaging core) which is equal to zero for $\xi \geq 1$ and $\int\limits_{|\xi|<1} \omega(\xi)\, d\xi = 1$; $R^n_\varepsilon(f \mid y)$ is a smoothing operator which is linear in f, $y = (x, t, \rho, s_0, \ldots, s_m) \in \mathbb{R}^n$ $(n = m+5)$. The coefficient $\gamma = \gamma[x, \boldsymbol{s}(x,t)]$ in (1) is treated as a composite function which is continued over (x,t) into $\mathbb{R}^3$ and over $\boldsymbol{s}$ into $\mathbb{R}^{m+1}$, and it is assumed that $\gamma(x,t,\varepsilon) = R^3_\varepsilon \gamma[x, \boldsymbol{s}(x,t)]$. In Eq. (2), the averaging operation is also applied to the coefficients $\rho(x,t)$ and $\boldsymbol{u}(x,t)$ continued into $\mathbb{R}^3$: $\rho(x,t,\varepsilon) = R^3_\varepsilon(\rho)$, and $\boldsymbol{u}(x,t,\varepsilon) = R^3_\varepsilon(\boldsymbol{u})$. We note that because the smoothing operator $R^n_\varepsilon(f)$ is linear, the properties (i) of the diffusion matrices $D = \{\lambda_{ij}\}$ $[(i,j) = \overline{1,m}]$ are retained for $D(\varepsilon) = \{R^n_\varepsilon(\lambda_{ij})\}$. Finally, we complete the regularization of problem (1)–(4) by changing the diagonal elements in $D(\varepsilon)$ assuming that

$$\lambda_{ii}(y,\varepsilon) = R^n_\varepsilon(\lambda_{ii}) + \varepsilon, \qquad i = \overline{1,m}.$$

Then in (i), we have $\sum\limits_{i=1}^{m} \lambda_{ij}(y,\varepsilon) = \varepsilon$, $i \neq j$ and denote the thus changed assumptions (i) by $(\mathrm{i})_\varepsilon$. We omit the argument ε in the coefficients of the regularized problem (1)–(4), considering them fairly smooth functions.

We first prove the solvability of the regularized problem (2), (4) for $\boldsymbol{s}$, assuming that one of the sets Γ^1 or Γ^2 is empty, i.e., (4) is the first $(\Gamma^2 = \varnothing)$ or second $(\Gamma^1 = \varnothing)$ initial-boundary-value problems. Then, sequentially considering Eqs. (2) for $s_k(x,t)$ and taking into account the quasitriangular nature of the augmented diffusion matrix $D_0 = \{\lambda_{ij}\}$, $(i,j) = \overline{0,m}$ [assumptions (ii)] due to choice of a small parameter $\delta \ll 1$, we obtain Schauder's estimates for $\boldsymbol{s}(x,t)$ in the space $H^{2+\alpha}(Q) = C^{2+\alpha, 1+\alpha/2}_{x,t}(Q)$ [8, p. 364]

$$\left| \boldsymbol{s} \right|^{(2+\alpha)}_Q \equiv \|\boldsymbol{s}\|_{H^{2+\alpha}(Q)} \leqslant M(\varepsilon). \tag{7}$$

Generally, for $\Gamma^k \neq \varnothing$ $(k = 1,2)$ estimate (7) becomes

$$\left| \boldsymbol{s} \right|^{(2+\alpha)}_{Q_\tau} \leqslant M(\varepsilon, \tau), \qquad Q_\tau = Q \backslash \mathcal{O}_\tau(\Gamma^1 \cap \Gamma^2), \tag{8}$$

where $\mathcal{O}_\tau$ is the τ-neighborhood of the conjugation lines $\Gamma^1 \cap \Gamma^2$ of the boundary conditions in (4).

We substitute the solution $\boldsymbol{s}(x,t)$ of the regularized problem (2), (4) into the coefficient $\gamma(x, \boldsymbol{s})$ of Eq. (1) and consider the regularized problem (1), (3). For the solution $(\boldsymbol{u}, p, \rho)$ of this problem, we obtain

$$\left| \boldsymbol{u} \right|^{(2+\alpha)}_Q + \left| \nabla p \right|^{(\alpha)}_Q + \left| \rho \right|^{(1)}_Q \leqslant M_0(\varepsilon), \qquad \left| (\rho, \rho_t, \nabla\rho) \right|^{(\alpha)}_Q \leqslant M_0. \tag{9}$$

The first of the estimates (9) is directly proved in [1, p. 133], and the second is proved there in the weaker form for $\rho(x,t) \in C^1(Q)$. Hölder's continuity of the functions $(\rho_t, \rho_{x_1}, \rho_{x_2})$ follows from the reasoning for the function $\rho(x,t)$ given in [1, p. 130]. Estimates (7)–(9) allow one, using Schauder's theorem, to prove that the regularized problem (1)–(4) has a solution $(\boldsymbol{s}, \boldsymbol{u}, p, \rho)$.

Since, physically, the functions $s_k(x,t)$ $(k = \overline{1,m})$ are the concentrations of the components of an inhomogeneous fluid, they should satisfy the following relations:

$$\sum_{k=1}^{m} s_k(x,t) = 1; \qquad 0 \leqslant s_k(x,t) \leqslant 1, \quad k = \overline{1,m}. \tag{10}$$

To prove (10), we combine both parts of Eqs. (2) for $s_k(x,t)$ $(k = \overline{1,m})$ and, taking into account condition $(\mathrm{i})_\varepsilon$, we obtain the parabolic equation $\rho\, ds/dt = \varepsilon \Delta s$ for the sum of concentrations $s = \sum\limits_{k=1}^{m} s_k$. Since $s|_{\Omega_0 \cup \Gamma^1} = 1$ and

$\nabla s \cdot \boldsymbol{n}|_{\Gamma^2} = 0$ [condition (6)], we have $s(x,t) = 1$ [$(x,t) \in Q$] according to the maximum principle. Then, by virtue of the equality $\sum\limits_{k=1}^{m-1} s_k = 1 - s_m$, Eq. (2) for $s_m(x,t)$ is written as

$$\rho \frac{ds_m}{dt} - \nabla \cdot \Big(\sum_{j=0}^{m} \bar{\lambda}_{mj} \nabla s_j \Big) = h_m,$$

where $\bar{\lambda}_{m0} = \lambda_{m0}$, $\bar{\lambda}_{mj} = 0$ ($j = \overline{1, m-1}$), and $\bar{\lambda}_m = \lambda_{m-1} - \sum\limits_{k=1}^{m-1} \lambda_{km}$, i.e., formally we can set $\lambda_m = \lambda_{m-1}$ in (2). We continue the coefficients λ_{ij} and h_i [$(i,j) = \overline{1,m}$] for $s_k \leqslant 0$ ($k = \overline{1,m}$) by their extreme values:

$$(\lambda_{ij}, h_i)\Big|_{s_k<0} = 0, \quad i \neq j, \qquad \lambda_{ii}\Big|_{s_k<0} = \lambda_{ii}(x,0).$$

Assuming that $\min s_k(x,t) < 0$ is reached at a certain internal point $(x_k, t_k) \in Q$ ($k = \overline{1,m}$), we arrive at inconsistency with the maximum principle since in the neighborhood of this point, the equation for $s_k(x,t)$ is homogeneous with respect to the derivatives: $\rho\, ds_k/dt - \nabla \cdot (\lambda_k \nabla s_k) = 0$.

Thus, the validity of relation (10) is established.

1.4. *Existence Theorem.* For the solution $(\boldsymbol{u}, p, \rho, \boldsymbol{s})$ of the regularized problem (1)–(4), we establish the validity of estimates of the form (7)–(9) with constants independent of the regularization parameter ε.

Let us define the following spaces of the functions $\boldsymbol{v}(x,t)$ and $(x,t) \in Q$ and the norm in them:

$$B^{k,k+1}(Q) = L_\infty(0,T; J^k(\Omega)) \cap L_2(0,T; J^{k+1}), \qquad k = 0,1,$$

$$J^k(\Omega) = \{\boldsymbol{v} \in W_2^k(\Omega), \operatorname{div} \boldsymbol{v} = 0\}, \qquad k = 0,1,2 \quad (W_2^0 \equiv L_2),$$

$$\|\boldsymbol{v}\|_Q^{(k,k+1)} = \sup \|\boldsymbol{v}\|_\Omega^{(k)} + \int_0^T (\|\boldsymbol{v}\|_\Omega^{(k+1)})^2\, dt, \qquad k = 0,1,$$

$$\|\boldsymbol{v}\|_\Omega^{(k)} = \|\boldsymbol{v}\|_{W_2^k(\Omega)}, \qquad k = 0,1,2.$$

For the other spaces, we adopt the standard notation of the norms from [1, 8].

For the solution of problem (1), (3) for $\gamma = 0$, Antontsev and Kazhikhov [1, chapter III] proved the following estimates, which are valid for $0 \leqslant \gamma \leqslant M_0 < \infty$:

$$\|\boldsymbol{u}\|_Q^{(1,2)} \leqslant M, \qquad \|(\nabla p, \boldsymbol{u}_t)\|_{2,Q} \leqslant M, \qquad \big|\rho\big|_Q^{(\beta)} \leqslant M, \qquad \beta > 0. \tag{11}$$

Here the constant M in (11) depends on $\|\boldsymbol{U}\|_Q^{(1,2)}$, $\|\boldsymbol{U}_t\|_{2,Q}$, $\|\rho^0\|_{C^1(\Omega)}$, $\|\boldsymbol{f}\|_{2,Q}$ and $\sup|\gamma|$.

We now address problem (2), (4) for the vector $\boldsymbol{s}(x,t)$ and first consider the case where $\Gamma^2 = \varnothing$ or $\Gamma^1 = \varnothing$. The well-known results for the first and second initial-boundary-value problems (2), (4) lead to the following estimates [8, p. 364]:

$$\big|\boldsymbol{s}\big|_Q^{(\alpha)} \leqslant M_1, \qquad \|\boldsymbol{s}\|_{q,Q}^{(2)} \leqslant M_1, \qquad \alpha > 0, \qquad q > 2. \tag{12}$$

Here $\|\cdot\|_{q,Q}^{(2)} = \|\cdot\|_{W_q^{2,1}(Q)}$, and the constant M_1 depends from M_0 in (10), $\|\boldsymbol{S}, \boldsymbol{G}\|_Q^{(2)}$, and $\sup|h_k|$.

The second estimate in (12) is obtained using the standard unity partition method and by considering the equations for $s_i(x,t)$ with the "frozen" coefficients $\lambda_{ii}(x_k, t_k)$; $(x_k, t_k) \in Q_k$, where $Q_k \subset Q$ is an elementary volume [1, pp. 234–235].

Reverting to problem (1), (3) with the functions $\boldsymbol{f}[x,t,\boldsymbol{s}(x,t)]$ and $\gamma[x, \boldsymbol{s}(x,t)] \in H^\alpha(Q) \equiv C^{\alpha,\alpha/2}(Q)$, we arrive at the estimates [1, p. 132]

$$\big|\boldsymbol{u}\big|_Q^{(2+\alpha)} + \big|\nabla p\big|_Q^{(\alpha)} + \big|\rho\big|_Q^{(1+\alpha)} \leqslant M_2, \tag{13}$$

where M_2 depends on M_1 in (12), $\big|\boldsymbol{U}\big|_Q^{(2+\alpha)}$, $\big|\rho^0\big|_\Omega^{(1+\alpha)}$, and $\big|\boldsymbol{f}\big|_Q^{(\alpha)}$. By virtue of (13), for problem (2), (4) we obtain

$$\big|\boldsymbol{s}\big|_Q^{(2+\alpha)} \leqslant M_3, \tag{14}$$

where M_3 is a function of M_2 and $\big|\boldsymbol{S}, \boldsymbol{G}\big|_Q^{(2+\alpha)}$.

Generally, when $\Gamma^k \neq \varnothing$ ($k = 1, 2$), i.e., (2), (4) is a mixed initial-boundary-value problem, proceeding as in Sec. 1.3, we obtain

$$\left|\boldsymbol{s}\right|_{Q_\tau}^{(2+\alpha)} \leqslant M_5(\tau), \qquad Q_\tau = Q \backslash \mathcal{O}_\tau(\Gamma^1 \cap \Gamma^2), \tag{15}$$

which for $(\boldsymbol{u}, p, \rho)$ implies the estimate

$$\left|\boldsymbol{u}\right|_{Q_\tau}^{(2+\alpha)} + \left|(\nabla p, \rho_t, \nabla\rho)\right|_{Q_\tau}^{(\alpha)} \leqslant M_6(\tau). \tag{16}$$

Passing to the limit as $\varepsilon \to 0$ for the regularization parameter ε, we arrive at the following statement.

Theorem 1. *Let assumptions* (i), (ii), (5), and (6) *be satisfied. Then if* $\Gamma^2 = \varnothing$ *or* $\Gamma^1 = \varnothing$*, there is a classical solution* $(\boldsymbol{u}, p, \rho, \boldsymbol{s}) \in H^{2+\alpha} \times H^{\alpha} \times H^{1+\alpha} \times H^{2+\alpha}$ *of problem* (1)–(4)*. If* $\Gamma^k \neq \varnothing$*,* $k = 1, 2$*, the solution of problem* (1)–(4) *also exists and* $\boldsymbol{u} \in B^{1,2}(Q) \cap H^{2+\alpha}(Q_\tau)$*,* $\nabla p \in H^{\alpha}(Q)$*,* $\rho \in H^{\alpha}(Q) \cap H^{1+\alpha}(Q_\tau)$*,* $\boldsymbol{s} \in H^{2+\alpha}(Q_\tau)$*, and* $\alpha > 0$.

2. Filtration of an Inhomogeneous Incompressible Fluid in a Porous Medium. 2.1. *Formulation of the Problem.* To describe the plane filtration process, we use the following model proposed in [3, 7]:

$$-\boldsymbol{u} = K\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0; \qquad \sigma\rho_t + \boldsymbol{u}\nabla\rho = 0. \tag{17}$$

Here $K(x, \boldsymbol{s}) = K_0(x)\mu^{-1}(\boldsymbol{s})$ [K_0 is the positively defined filtration tensor; $\mu = \exp\Big(\sum\limits_{i=1}^{m} s_i \ln \mu_i\Big)$, and $\mu_i(\boldsymbol{s})$ are the dynamic viscosities of the components] and $\sigma(x)$ is the porosity coefficient. As in system (1), the last equality is the condition of incompressibility of the mixture and the relation $\nabla \cdot \boldsymbol{u} = 0$ is a consequence of this condition and the flow continuity equation $\sigma\rho_t + \nabla \cdot (\rho\boldsymbol{u}) = 0$. The first relation in (17) is generalized Darcy's law. The temperature s_0 and the concentrations $s_i = \rho_i/\rho$ ($i = \overline{1, m}$) of the mixture components satisfy Eqs. (2), whose coefficients obey assumptions (i) and (ii).

The curve $\partial\Omega$ is divided into adjacent arcs l_i^1 and l_i^2 ($i = \overline{1, n}$) and, in accordance with this, the surface of the cylinder $\Gamma = \partial\Omega \times (0, T)$ consists of the sets $\Lambda^k = \cup_1^n l_i^k \times (0, T)$ ($k = 1, 2$) and $\Gamma = \Lambda^1 \cup \Lambda^2$, and, generally speaking, Λ^k do not coincide with Γ^k in (4). The boundary conditions (4) for the vector $\boldsymbol{s}$ are conserved, and conditions (3) for $(\boldsymbol{u}, p, \rho)$ are replaced by the following:

$$p\Big|_{\Lambda^1} = p_0, \qquad \boldsymbol{u} \cdot \boldsymbol{n}\Big|_{\Lambda^2} = -U; \qquad (\rho - \rho^0)_{\partial\Omega} = 0. \tag{18}$$

If $\Lambda^1 = \varnothing$, we additionally require that $\int\limits_\Gamma p\,dx\,dt = \int\limits_\Gamma U\,dx\,dt = 0$.

2.2. *Solvability of the Filtration Problem.* We apply the regularization of Sec. 1.3 to problem (2), (4) for the vector $\boldsymbol{s}(x, t)$ and perform Steklov's averaging of the augmented coefficients (K, ρ, σ) of Eqs. (17) and the boundary functions (p_0, ρ^0, U) in (18). The classical solvability of the regularized compatible problem (2), (4), (17), (18) follows, according to Schauder's theorem, from the validity of the following estimates for its solutions $(\boldsymbol{s}, \rho, p)$:

$$|\boldsymbol{s}|_{Q_\tau}^{(2+\alpha)} \leqslant M(\varepsilon), \qquad |\rho|_{Q^\tau}^{(1+\alpha)} \leqslant M(\varepsilon), \qquad |p|_{Q^\tau}^{(2+\alpha)} \leqslant M(\varepsilon). \tag{19}$$

Here $Q^\tau = Q \setminus Q^\tau(\Lambda^1 \cap \Lambda^2)$ (Q^τ is the τ-neighborhood $\Lambda^1 \cap \Lambda^2$) and $Q_\tau = Q \setminus O_\tau(\Gamma^1 \cap \Gamma^2)$).

If Λ^1 or Λ^2 are empty, then $Q_\tau = Q$ in (19). The first and second estimates (19) coincide, according to (15) and (16), and the third estimate is established in [1, p. 239].

To prove estimates of the form (19) that are uniform in ε, we first consider the regularized problem (17), (18) for $(\boldsymbol{u}, p, \rho)$.

Let $\partial\Omega \subset C_*^{2+\alpha}$ [1, p. 231], i.e., at each point x that does not belong to $\Lambda^1 \cap \Lambda^2$, the boundary is locally straightened, and in the neighborhood of the points $x \in \Lambda^1 \cap \Lambda^2$, it is locally mapped onto the right angle. We assume that

$$\ln(K\boldsymbol{\xi}, \boldsymbol{\xi}) + \left|\ln(\sigma, \rho^0)\right|_{\Omega}^{(\alpha)} + \|(D^1 p_0, U)\|_{q,\partial\Omega} \leqslant N_1 < \infty, \tag{20}$$

where $|\boldsymbol{\xi}| = 1$, $q > 2$, and D^k are the derivatives along $\partial\Omega$. Then, according to [1, p. 260], we have

$$\|\nabla p\|_{q,\Omega} + \left|p\right|_{\Omega}^{(\alpha)} \leqslant N_2(N_1), \qquad (q, \alpha) > 0; \qquad |\ln\rho| \leqslant N_1. \tag{21}$$

Reverting now to problem (2), (4) for $\boldsymbol{s}$ and proceeding as in Sec. 1.4, we obtain

$$\|\boldsymbol{s}\|_{q,Q_\tau}^{(1,0)} + \left|\boldsymbol{s}\right|_{Q_\tau}^{(\alpha)} \leqslant N_3(N_2) \qquad (q > 2,\ \alpha > 0), \tag{22}$$

In this case, if Γ^1 or Γ^2 are empty, then $Q_\tau = Q$.

Let, in addition to (19), the following inequality be satisfied:

$$|(K_x, K_{s_0}, \dots, K_{s_m})| + \|(D^2 p_0, D^1 U)\|_{q,\partial\Omega} + \left|(\sigma, \nabla\rho^0)\right|_{\Omega}^{\alpha} \leqslant N_4, \tag{23}$$

where $q > 2$ and $\alpha > 0$. Estimates (21) and (22) allow an increase in the smoothness of the solution of problem (17), (18):

$$\|\boldsymbol{u}\|_{q,\Omega}^{(1)} + \left|\boldsymbol{u}\right|_{Q_\tau}^{(\alpha)} + \left|\rho\right|_{Q_\tau}^{(\alpha)} \leqslant N_5(N_4) \qquad (q > 2,\ \alpha > 0), \tag{24}$$

and $Q_\tau = Q$ if one of Γ^k $(k = 1, 2)$ is empty. Inequalities (24) for $\boldsymbol{u}(x,t)$ are established in [1, p. 236], and those for $\rho(x,t)$ in [1, p. III]. Taking into account (24), for the solution $\boldsymbol{s}(x,t)$ of problem (2), (4), we obtain estimates of the form (19) with the constant factor independent of ε:

$$\left|\boldsymbol{s}\right|_{Q_\tau}^{(2+\alpha)} \leqslant N_6 \qquad (\alpha > 0). \tag{25}$$

Here $Q_\tau = Q$, if $\Gamma^1 = \varnothing$ or $\Gamma^2 = \varnothing$.

For arbitrary functions $\varphi(x) \in W_2^1(\Omega)$ $(\varphi|_{\Lambda^1} = 0)$ and $\psi(x,t) \in W_2^1(Q)$ $(\psi|_{\partial Q} = 0)$, we introduce the integral identities

$$(\boldsymbol{u}, \nabla\varphi)_\Omega = (U, \varphi)_{\Lambda^2}, \qquad (\sigma\rho, \psi_t)_Q + (\boldsymbol{u}\rho, \nabla\psi)_Q = 0, \tag{26}$$

where $(f,g)_E = \int\limits_E fg\, dE$.

Passing to the limit as $\varepsilon \to 0$ in problem (2), (4) for $\boldsymbol{s}(x,t,\varepsilon)$ and in the integral identities (26) for $\boldsymbol{u}(x,t,\varepsilon)$, $\rho(x,t,\varepsilon)$, we arrive at the following statement.

Theorem 2. *Let assumptions* (i) (ii), (5), and (6) *be satisfied for problem* (2), (4) *and assumptions* (19) and (23) *for problem* (17), (18). *Then, the compatible problem* (2), (4), (17), (18), *which describes the filtration process of an inhomogeneous incompressible viscous fluid in porous media, has at least one solution* $(\boldsymbol{u}, \rho, \boldsymbol{s})$ *that satisfies equalities* (2) and (4) *for* $\boldsymbol{s}$, *the integral identities* (26), *and the conditions* $(p - p_0)_{\Lambda^1} = (\rho - \rho^0)_\Omega = 0$ *for* $(\boldsymbol{u}, \rho)$. *In this case,* $\boldsymbol{u}(x,t) \in L_\infty[0,T; W_q^1(\Omega)] \cap H^\alpha(Q_\tau)$, $\rho(x,t) \in H^\alpha(Q_\tau)$, $\boldsymbol{s}(x,t) \in H^{2+\alpha}(Q_\tau)$, $q > 2$, *and* $\alpha > 0$. *If* $\Gamma^1 = \varnothing$ *or* $\Gamma^2 = \varnothing$, *then* $Q_\tau = Q$.

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